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Nonlinear $\mathcal{H}_\infty$ Control Designs with Axisymmetric Spacecraft Control

Qian Zheng* and Fen Wu†
North Carolina State University, Raleigh, North Carolina 27695

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In this paper, we study nonlinear control of a spacecraft symmetric about its principal axis with two control torques. Using a computationally efficient $\mathcal{H}_\infty$ control design procedure, attitude stabilization and command tracking problems of the axisymmetric spacecraft are solved locally. The proposed nonlinear $\mathcal{H}_\infty$ control approach uses higher order Lyapunov functions and reformulates the difficult Hamilton–Jacobian–Isaacs inequalities as semidefinite optimization conditions. Sum-of-squares programming techniques are then applied to obtain computationally tractable solutions, from which nonlinear control laws will be constructed. The nonlinear $\mathcal{H}_\infty$ control designs for spacecraft are capable of exploiting the most suitable forms of Lyapunov functions for performance improvement.

I. Introduction

SPACECRAFT attitude dynamics has inherent nonlinear characteristics such as cross-coupling gyroscopic terms that cannot be ignored. Moreover, increasingly stringent operating requirements of spacecraft demand high performance flight control systems. For example, future commercial and military spacecraft are envisioned to have large angle maneuver capability and high pointing accuracy. In addition, limited propellant on board necessitates optimal control strategy for long term spacecraft operations. These capabilities and operating requirements are imperative for the demanding tasks of future space missions, such as very large base interferometry, formation flying, and autonomous shuttle docking and servicing.

Stabilizing feedbacks of a spacecraft with or without disturbance inputs have been studied for a long time. Because of a significant cross-coupling effect of spacecraft, linear control designs based on its linearized dynamics are often inadequate [1]. Wie and Barba [2] derived a number of simple nonlinear control schemes using quaternion and angular velocity feedback. Tsiotras [3] extended these results and designed different linear/nonlinear controllers by proper selection of Lyapunov functions. These control laws could recover the optimal cost involving both attitude and velocity asymptotically but may lead to high-gain controllers. Generally speaking, the synthesis of optimal feedback control for a rigid body spacecraft remains a difficult problem and was mainly addressed in the context of time/fuel optimal maneuvers [4]. The main obstruction in obtaining optimal control laws is to efficiently solve the Hamilton–Jacobi equation, especially when the cost includes a penalty term on the control effort. To circumvent this difficulty, an inverse optimal control approach was proposed in [5] that leads to a controller optimal with respect to a cost functional determined a posteriori. Nevertheless, it requires the knowledge of a control Lyapunov function and a stabilizing control law of a particular form. Attitude control of an orbiting spacecraft in the presence of uncertainty was also addressed in [6] using sliding mode control. Nonlinear $\mathcal{H}_\infty$ control has been used in [7] for spacecraft control. It, however, solved the Hamilton–Jacobian–Isaacs (HJI) inequality with a special form of Lyapunov functions, thus only providing an upper bound of the performance. The aforementioned techniques mainly are Lyapunov-based control designs. At the current stage, a serious drawback of these designs is the lack of a systematic procedure in constructing such Lyapunov functions.

Other than spacecraft stabilization problems, another issue highly relevant to spacecraft operation is the tracking of prescribed attitude trajectories. Adaptive control and sliding mode control were proposed in [8,9] for the spacecraft tracking problem. However, the full states have to be measured for these control algorithms and no robustness is guaranteed in the presence of unmodeled dynamics. On the other hand, a passivity-based control approach was proposed in [10] to achieve rigid body attitude regulation without velocity measurement. The second approach to avoid the use of velocity information was based on nonlinear observers [11]. It is noted that these output feedback controllers do not have desired optimality and robustness properties.

The contributions of this paper are twofold: As the first contribution, we will present an output feedback $\mathcal{H}_\infty$ synthesis condition in terms of HJI inequalities for a class of polynomial nonlinear systems without orthonormal and decoupling assumptions on the plant structure. A computationally efficient approach based on sum-of-squares (SOS) programming [12,13] will then be proposed to overcome the difficulty in solving generalized HJI inequalities, which also facilitate synthesizing higher order Lyapunov functions and their associated nonlinear control laws. In comparison, a Taylor series approximation of Lyapunov functions was first proposed in [14] to solve HJI equations term by term for state feedback $\mathcal{H}_\infty$ control. Taylor series expansion was also considered in [15] to find storage functions of the HJI equation by iterating over control policy u. The difficulty with series approximation is, however, to guarantee stability of the closed-loop system for finite truncations of the series. On the other hand, a Galerkin successive approximation was proposed in [16] to reduce HJI equations to a sequence of linear partial differential equations. Then the Galerkin method was applied to solve control input and worst case disturbance iteratively to find an approximated solution. As a drawback, its computational cost increases exponentially as the number of states increase.

The second contribution is to demonstrate the proposed $\mathcal{H}_\infty$ control design approach to underactuated spacecraft control. Equipped with an effective nonlinear $\mathcal{H}_\infty$ control design tool, we will study both of the attitude stabilization and command tracking problems for an axisymmetric rigid spacecraft with two controls. Accurate control of the axisymmetric spacecraft has practical importance as it is required during deployment and station keeping of spacecraft in orbit. Recently, an SOS-based state feedback stabilizing controller has been developed in [17] for a spacecraft large angle maneuver with promising results. In our work, a state

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*Ph.D. Student, Department of Mechanical and Aerospace Engineering; qzheng@unity.ncsu.edu.

†Associate Professor, Department of Mechanical and Aerospace Engineering; fwu@eos.ncsu.edu. Associate Fellow AIAA.

feedback $\mathcal{H}_\infty$ controller will be designed to stabilize the axisymmetric spacecraft on a rotating plane and will be compared with existing nonlinear control design in [18]. Moreover, we will develop a nonlinear output feedback $\mathcal{H}_\infty$ controller for spacecraft tracking some desired trajectory in the presence of external disturbances. The proposed output feedback designs will be advantageous by enhancing control design flexibility and reducing onboard sensor hardware.

The notation used in this paper is fairly standard. $\mathcal{Z}$ represents a positive integer set. $\mathbf{R}$ stands for the set of real numbers and $\mathbf{R}_+$ for the nonnegative real numbers. $\mathbf{R}^{m\times n}$ is the set of real $m \times n$ matrices. We use $\mathbf{S}^{n\times n}$ to denote real, symmetric $n \times n$ matrices, and $\mathbf{S}_+^{n\times n}$ for positive definite matrices. In large symmetric matrix expressions, terms denoted by $\star$ are induced by symmetry. Given a vector

$$x = [\,x_1 \quad \cdots \quad x_n\,]^T \in \mathbf{R}^n, \qquad \frac{\partial V}{\partial x} = \left[\,\frac{\partial V}{\partial x_1} \quad \frac{\partial V}{\partial x_2} \quad \cdots \quad \frac{\partial V}{\partial x_n}\,\right]$$

is the derivative of V with respect to x. $M^{[r]}(x)$ is a monomial vector of x with all components having degree no more than r. $\|x\|_2$ is the $\mathcal{L}_2$ norm of x. A multivariate polynomial $p(x)$ is an SOS if there exist polynomials $p_1(x), \ldots, p_\ell(x)$ such that

$$p(x) = \sum_{i=1}^{\ell} p_i^2(x)$$

We will denote the set of SOS polynomials as Φ_{SOS}.

This paper is organized as follows: Section II generalizes nonlinear $\mathcal{H}_\infty$ synthesis conditions by removing orthonormal and decoupling assumptions. It also provides the design procedure for the nonlinear output feedback $\mathcal{H}_\infty$ control problem using the SOS programming approach. In Sec. III, we first introduce the nonlinear model of an axisymmetric spacecraft. The remainder of Sec. III is devoted to nonlinear $\mathcal{H}_\infty$ control designs for the spacecraft to achieve attitude stabilization and command tracking objectives. Simulation results are also provided to demonstrate the advantages of new $\mathcal{H}_\infty$ control designs and compare with existing nonlinear spacecraft control laws. Finally, the paper concludes in Sec. IV.

II. Nonlinear $\mathcal{H}_\infty$ Synthesis Using Sum-of-Squares Programming

A. Background on Sum-of-Squares Programming

Consider a vector $x = [\,x_1 \quad \cdots \quad x_n\,]^T \in \mathbf{R}^n$ and the multi-index vector $k = [\,k_1 \quad \cdots \quad k_n\,]^T \in \mathbf{Z}^n$ with

$$\bar{k} = \sum_{i=1}^{n} k_i$$

For integer $\bar{k}$, we will define

1) $m_{\bar{k}}(x)$: monomial of x with degree $\bar{k}$

$$m_{\bar{k}}(x) = x^{\bar{k}} = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}, \qquad \sum_{i=1}^{n} k_i = \bar{k}$$

2) $f(x)$: polynomial function of x composed of combination of monomials with coefficients $c_{\bar{k}}$'s

$$f(x) = \sum_{\bar{k}} c_{\bar{k}} m_{\bar{k}}(x)$$

A multivariate polynomial $f(x_1, \ldots, x_n)$ is an SOS, if there exist polynomials $f_1(x), \ldots, f_m(x)$ such that

$$f(x) = \sum_{i=1}^{m} f_i^2(x) \tag{1}$$

This clearly implies $f(x) \geq 0$ for any $x \in \mathbf{R}^n$. SOS decomposition provides a sufficient condition for verifying nonnegativity of a multivariate polynomial. Moreover, the SOS decomposition condition (1) is equivalent to the existence of a positive semidefinite matrix G such that

$$f(x) = M^T(x) G M(x)$$

where $M(x)$ is some vector of monomials and G is called a Gramian matrix. Therefore, checking if a given multivariate polynomial $f(x)$ is an SOS is related to the nonnegativity of a Gramian matrix associated with the polynomial [12], which can be solved efficiently as a semidefinite programming problem [13].

The main advantages of SOS programming are the resulting computational tractability and the algorithmic character of its solution procedure [12]. This could help provide coherent methodology of synthesizing Lyapunov functions for nonlinear systems. In addition, the importance of the SOS technique also lies in its ability to provide tractable relaxations for many difficult optimization problems. As a powerful and promising technique, SOS programming provides an efficient way to exploit polynomial nonlinear systems, which can be used to approximate many nonlinear systems. Recent studies have shown the strength of SOS programming in solving nonlinear analysis [19–21] and nonlinear control synthesis [22–24] problems. It will be used in this study to solve the $\mathcal{H}_\infty$ control problem for axisymmetric spacecraft with higher order Lyapunov functions.

B. Nonlinear Output Feedback $\mathcal{H}_\infty$ Control Problem

Now, we consider a polynomial nonlinear system that is affine in exogenous disturbance and control input

$$\begin{cases} \dot{x} = A(x)x + B_1(x)d + B_2(x)u \\ e = C_1(x)x + D_{12}u \\ y = C_2(x)x + D_{21}d \end{cases} \tag{2}$$

where the system state $x \in \mathbf{R}^n$, control input $u \in \mathbf{R}^{n_u}$, exogenous disturbance $d \in \mathbf{R}^{n_d}$, controlled output $e \in \mathbf{R}^{n_e}$, and measured output $y \in \mathbf{R}^{n_y}$. All of the state-space entries are polynomial functions of the state x with compatible dimensions. For simplicity, we also assume that D_{12} and D_{21} are constant matrices with full column and row ranks, respectively. Most of the previous literature [25–27] derived the nonlinear output feedback $\mathcal{H}_\infty$ synthesis condition by enforcing the plant with orthonormal and decoupling structures such as

$$D_{12}^T [\,C_1(x) \quad D_{12}\,] = [\,0 \quad I\,] \qquad \begin{bmatrix} B_1(x) \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

These assumptions are quite restrictive and will be removed in this study.

The objective of the nonlinear output feedback $\mathcal{H}_\infty$ control is to design an output feedback controller such that the nonlinear closed-loop system is asymptotically stable and its $\mathcal{L}_2$ gain from d to e is less than γ, that is,

$$\|e\|_2 \leq \gamma \|d\|_2 \quad \text{when } x(0) = 0$$

A standard nonlinear $\mathcal{H}_\infty$ control problem was solved in [27] by decomposing output feedback control into full information and full control problems. Its solvability condition was given in the form of two HJI inequalities, which are nonlinear partial differential inequalities and extremely difficult to solve. In the following, we will present the $\mathcal{H}_\infty$ synthesis condition for nonlinear system (2) with relaxed plant assumptions. This relaxation would expand the nonlinear $\mathcal{H}_\infty$ control framework to a wider class of physical systems. Moreover, it would help provide additional freedom to search for a suitable Lyapunov function and result in better nonlinear $\mathcal{H}_\infty$ control laws to meet the design objective.

Theorem 1: Given a scalar $\gamma > 0$, the nonlinear output feedback $\mathcal{H}_\infty$ control problem is (locally) solvable for the nonlinear system (2) if there exist two positive definite matrix functions $U(x)$ and $V(x)$, satisfying $U(0) = V(0) = 0$ such that

$$\mathcal{H}_{FI}(U,\gamma,x) < 0 \tag{3}$$

$$\mathcal{H}_{FC}(V,\gamma,x) - \gamma^{-2}\mathcal{H}_{FI}(U,\gamma,x) < 0 \tag{4}$$

$$\frac{\partial^2}{\partial x^2}(\mathcal{H}_{FC} - \gamma^{-2}\mathcal{H}_{FI})|_{x=0} < 0 \tag{5}$$

$$W(x) := V(x) - \gamma^{-2}U(x) \geq 0 \tag{6}$$

where

$$\begin{aligned}
&\mathcal{H}_{FI}(U,\gamma,x)\\
&:= \frac{\partial U}{\partial x}A(x)x + x^T C_1^T(x)[I - D_{12}(D_{12}^T D_{12})^{-1}D_{12}^T]C_1(x)x\\
&- \frac{1}{2}\frac{\partial U}{\partial x}B_2(x)(D_{12}^T D_{12})^{-1}D_{12}^T C_1(x)x\\
&- \frac{1}{2}x^T C_1^T(x)D_{12}(D_{12}^T D_{12})^{-1}B_2^T(x)\frac{\partial U^T}{\partial x}\\
&+ \frac{1}{4}\frac{\partial U}{\partial x}\left[\frac{1}{\gamma^2}B_1(x)B_1^T(x) - B_2(x)(D_{12}^T D_{12})^{-1}B_2^T(x)\right]\frac{\partial U^T}{\partial x}
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_{FC}(V,\gamma,x) &:= \frac{\partial V}{\partial x}A(x)x\\
&+ \frac{1}{4}\frac{\partial V}{\partial x}B_1(x)[I - D_{21}^T(D_{21}D_{21}^T)^{-1}D_{21}]B_1^T(x)\frac{\partial V^T}{\partial x}\\
&- \frac{1}{2}x^T C_2^T(x)(D_{21}D_{21}^T)^{-1}D_{21}B_1^T(x)\frac{\partial V^T}{\partial x}\\
&- \frac{1}{2}\frac{\partial V}{\partial x}B_1(x)D_{21}^T(D_{21}D_{21}^T)^{-1}C_2(x)x\\
&+ x^T\left[\frac{1}{\gamma^2}C_1^T(x)C_1(x) - C_2^T(x)(D_{21}D_{21}^T)^{-1}C_2(x)\right]x
\end{aligned}$$

Moreover, one of the nth-order nonlinear output feedback controllers is given by

$$\begin{cases} \dot{x}_c = A(x_c)x_c + B_1(x_c)F_1(x_c) + B_2(x_c)F_0(x_c) + L_0(x_c)[C_2(x_c)x_c + D_{21}F_1(x_c) - y] \\ u = F_0(x_c) \end{cases} \tag{7}$$

where the matrix functions $F_0(x)$, $F_1(x)$, and $L_0(x)$ are defined as

$$F_0(x) := -(D_{12}^T D_{12})^{-1}\left[\frac{1}{2}B_2^T(x)\frac{\partial U^T}{\partial x} + D_{12}^T C_1(x)x\right] \tag{8}$$

$$F_1(x) := \frac{1}{2\gamma^2}B_1^T(x)\frac{\partial U^T}{\partial x} \tag{9}$$

$$\frac{\partial W(x)}{\partial x}L_0(x) := -2[C_2(x)x + D_{21}F_1(x)]^T(D_{21}D_{21}^T)^{-1} \tag{10}$$

Note that generalized synthesis conditions (3) and (4) are also in the form of HJI inequalities. For nonlinear state feedback $\mathcal{H}_\infty$ control problem, the solvability condition degenerates to a single HJI inequality (3). In the special case when both $U(x)$ and $V(x)$ are quadratic functions of state x, conditions (3–6) become a set of state-dependent linear matrix inequalities. Nevertheless, with polynomial Lyapunov functions, the bilinear term

$$-\frac{1}{4}\frac{\partial U}{\partial x}B_2(x)(D_{12}^T D_{12})^{-1}B_2^T(x)\frac{\partial U^T}{\partial x}$$

will render condition (3) nonconvex. SOS programming [12] could provide an effective way to solve conditions (3) and (4) by reformulating them into SOS optimization problems. In the following sections, we will develop a systematic design procedure for nonlinear output feedback $\mathcal{H}_\infty$ control and obtain computationally tractable solutions for polynomial $U(x)$ and $V(x)$, that is,

$$U(x) = \frac{1}{2}M^{[n_p]}(x)^T P M^{[n_p]}(x), \quad n_p \geq 1$$
$$V(x) = \frac{1}{2}N^{[n_q]}(x)^T Q N^{[n_q]}(x), \quad n_q \geq 1$$

using SOS programming tools [13].

C. Iterative Algorithm to Solve Nonlinear State Feedback $\mathcal{H}_\infty$ Condition

For a quadratic Lyapunov function $U_0(x) = \frac{1}{2}x^T P_0 x > 0$ and a state feedback controller $u_0 = K_0(x)x$, the closed-loop plant becomes

$$\begin{cases} \dot{x} = [A(x) + B_2(x)K_0(x)]x + B_1(x)d \\ e = [C_1(x) + D_{12}K_0(x)]x \end{cases}$$

Because of the quadratic form of the Lyapunov function, it is sufficient to reformulate condition (3) as the following SOS optimization problem:

$$\min \gamma_0 \qquad \text{subject to } -v^T\mathcal{S}_{FI0}v \in \Phi_{\text{SOS}} \tag{11}$$

where v is a free vector of compatible dimension and

$$\mathcal{S}_{FI0} := \begin{bmatrix} \{\frac{1}{2}[A(x)R_0 + R_0A^T(x)] - \frac{1}{4}B_2(x)(D_{12}^T D_{12})^{-1}B_2^T(x) - \frac{1}{2}B_2(x)(D_{12}^T D_{12})^{-1}D_{12}^T C_1(x)R_0 - \frac{1}{2}R_0C_1^T(x)D_{12}(D_{12}^T D_{12})^{-1}B_2^T(x)\} & \star & \star \\ [I - D_{12}(D_{12}^T D_{12})^{-1}D_{12}^T]^{\frac{1}{2}}C_1(x)R_0 & -I & \star \\ B_1^T(x) & 0 & -4\gamma_0^2 I \end{bmatrix}$$

with $R_0 = P_0^{-1} > 0$. The optimization problem (11) can be solved using SOS programming techniques to obtain a quadratic Lyapunov function $U_0(x)$ with $\mathcal{H}_\infty$ performance γ_0. Consequently, an initial state feedback control that renders the closed-loop $\mathcal{H}_\infty$ norm less than γ_0 will be

$$u_0(x) = -\left(D_{12}^T D_{12}\right)^{-1}\left[B_2^T(x)R_0^{-1} + D_{12}^T C_1(x)\right]x$$

On the other hand, an equivalent SOS condition for state feedback $\mathcal{H}_\infty$ control problem will be

$$\min \gamma_i \qquad \text{subject to } U_i(x) \in \Phi_{\text{SOS}} \qquad -v^T\mathcal{S}_{FIi}v \in \Phi_{\text{SOS}} \tag{12}$$

where

$$\mathcal{S}_{FIi} := \begin{bmatrix} \left\{\frac{\partial U_i}{\partial x}[A(x)x + B_2(x)u_{i-1}(x)] + [C_1(x)x + D_{12}u_{i-1}(x)]^T[C_1(x)x + D_{12}u_{i-1}(x)]\right\} & \frac{\partial U_i}{\partial x}B_1(x) \\ B_1^T(x)\frac{\partial U_i^T}{\partial x} & -4\gamma_i^2 I \end{bmatrix}$$

For any fixed $u_{i-1}(x)$, conditions in Eq. (12) are convex about variables $U_i(x)$ and γ_i. Using a polynomial Lyapunov function $U_i(x)$, we will be able to achieve a better performance $\gamma_i < \gamma_{i-1}$ by solving a new SOS problem (12). It is clear that the optimization problem is always feasible [at least for $U_i(x) = U_{i-1}(x)$]. Then a new controller u_i can be computed from

$$u_i(x) = -(D_{12}^T D_{12})^{-1}\left[\frac{1}{2}B_2^T(x)\frac{\partial U_i^T}{\partial x} + D_{12}^T C_1(x)x\right] \tag{13}$$

and the next round of iteration will be continued by solving Eq. (12) repeatedly. The iterative process will stop when $|\gamma_i - \gamma_{i-1}|$ is sufficiently small.

Although we start from a quadratic Lyapunov function, the final Lyapunov function will be in polynomial forms of x. The resulting state feedback controller is then specified by Eq. (13) at the exit. This iterative algorithm will improve the closed-loop performance γ_i gradually as i increases and finally leads to a suboptimal solution for state feedback $\mathcal{H}_\infty$ control.

D. Design Procedure for Nonlinear Output Feedback $\mathcal{H}_\infty$ Control

Summarizing previous discussions, we will propose a control design procedure to solve the nonlinear output feedback H_∞ synthesis condition for polynomial nonlinear systems (2).

1) Using SOS programming and the iterative algorithm in Sec. II.C, the HJI inequality (3) will be solved to obtain a polynomial Lyapunov function

$$U(x) = \tfrac{1}{2}M^{[n_p]}(x)^T P M^{[n_p]}(x), \qquad n_p \geq 1$$

and a performance level $\gamma > 0$.

2) Calculate $F_0(x)$ and $F_1(x)$ by

$$F_0(x) = -(D_{12}^T D_{12})^{-1}\left[\frac{1}{2}B_2^T(x)\frac{\partial U^T}{\partial x} + D_{12}^T C_1(x)x\right]$$
$$F_1(x) := \tilde{F}_1(x)x = \frac{1}{2\gamma^2}B_1^T(x)\frac{\partial U^T}{\partial x}$$

3) Reformulate conditions (4–6) as the following SOS optimization problem:

$$\min \hat{\gamma} \qquad \text{subject to } V(x) \in \Phi_{\text{SOS}} \qquad -v_1^T \mathcal{S}_{FC} v_1 \in \Phi_{\text{SOS}}$$
$$-v_2^T\left[\frac{\partial^2}{\partial x^2}(\mathcal{H}_{FC} - \gamma^{-2}\mathcal{H}_{FI})\bigg|_{x=0}\right]v_2 \in \Phi_{\text{SOS}} \tag{14}$$
$$V(x) - \gamma^{-2}U(x) \in \Phi_{\text{SOS}}$$

where v_1 and v_2 are free vectors of compatible dimensions and

We solve optimization problem (14) by SOS programming to obtain polynomial Lyapunov function

$$V(x) = \tfrac{1}{2}N^{[n_q]}(x)^T Q N^{[n_q]}(x), \qquad n_q \geq 1$$

and a performance value $\hat{\gamma} \leq \gamma$.

4) Compute $L_0(x)$ from the equation

$$\frac{\partial W(x)}{\partial x}L_0(x) = -2[C_2(x)x + D_{21}F_1(x)]^T(D_{21}D_{21}^T)^{-1} \tag{15}$$

as

$$L_0(x) = -2T^{-1}(x)[C_2(x) + D_{21}\tilde{F}_1(x)]^T(D_{21}D_{21}^T)^{-1}$$

where $T(x)$ is a nonsingular polynomial function satisfying

$$\frac{\partial W}{\partial x} = x^T T(x)$$

5) Finally, the output feedback controller (7) will be constructed with a closed-loop $\mathcal{H}_\infty$ performance bounded by γ.

In the proposed control design procedure, we first reformulate synthesis conditions (3–6) into optimization problems (12) and (14). The resulting conditions are matrix inequalities with polynomial entries and linear about their decision variables. Using SOS programming techniques, both optimizations (12) and (14) are then solved with polynomial complexity. The closed-loop Lyapunov function is given by $U(x) + W(x - x_c)$, which could be any higher order polynomials. Note that $L_0(x)$ is not unique for a given $W(x)$. Since $W(x)$ has at least an order of 2, it is clear that $\partial W(x)/\partial x$ can be rewritten as $x^T T(x)$ for a nonsingular polynomial matrix $T(x)$ of dimension $n \times n$. Therefore, one solution of $L_0(x)$ satisfying Eq. (15) will be

$$L_0(x) = -2T^{-1}(x)[C_2(x) + D_{21}\tilde{F}_1(x)]^T(D_{21}D_{21}^T)^{-1}$$

It should be pointed out that Theorem 1 provides a local solution to the nonlinear output feedback $\mathcal{H}_\infty$ control problem. Most of the previous nonlinear control research considered a global stabilization problem. Nevertheless, it is not always possible to find a global stabilizing controller. Moreover, in a restricted region, local controllers will often perform better than global controllers. To solve the nonlinear $\mathcal{H}_\infty$ synthesis condition locally, the SOS optimization problems (12) and (14) should be modified by adding state region constraints. This will help improve solvability of the synthesis conditions through SOS programming and lead to optimized solutions for the nonlinear output feedback $\mathcal{H}_\infty$ control in a local region. These constraints also designate a state-space region (i.e., domain of validity) on which the synthesized nonlinear $\mathcal{H}_\infty$ control is valid.

$$\mathcal{S}_{FC} := \begin{bmatrix} \begin{matrix}\Big\{-\gamma^{-2}\mathcal{H}_{FI}(U,\gamma,x) + \frac{\partial V}{\partial x}A(x)x - x^T C_2^T(x)(D_{21}D_{21}^T)^{-1}C_2(x)x - \frac{1}{2}x^T C_2^T(x)(D_{21}D_{21}^T)^{-1}D_{21}B_1^T(x)\frac{\partial V^T}{\partial x} \\ -\frac{1}{2}\frac{\partial V}{\partial x}B_1(x)D_{21}^T(D_{21}D_{21}^T)^{-1}C_2(x)x\Big\}\end{matrix} & \star & \star \\ \frac{1}{2}[I - D_{21}^T(D_{21}D_{21}^T)^{-1}D_{21}]^{\frac{1}{2}}B_1^T(x)\frac{\partial V^T}{\partial x} & -I & \star \\ C_1(x)x & 0 & -\hat{\gamma}^2 I \end{bmatrix}$$

III. Nonlinear $\mathcal{H}_\infty$ Control Designs for Axisymmetric Spacecraft

A. Modeling of Axisymmetric Spacecraft

In this section, we will present an axisymmetric rigid body spacecraft model [18] with two control torques acting perpendicular to each other and its symmetry axis.

To describe spacecraft dynamics, two reference frames will be introduced (see Fig. 1 and [18,28]). The first one is the body-fixed reference frame $\mathbf{b} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$, which is located at the center of mass and aligned along the principal axes of axisymmetric spacecraft. In the body-fixed reference frame, denote the angular velocity vector and moment of inertia with respect to three body axes as $\omega_1, \omega_2, \omega_3$ and I_1, I_2, I_3. We assume that $\hat{b}_3$ is pointing along the symmetry axis and $I_1 = I_2$. Furthermore, two control torques T_1 and T_2 are generated along $\hat{b}_1$ and $\hat{b}_2$ axes and span a two-dimensional plane orthogonal to the axis of symmetry. The dynamics equations of spacecraft with respect to this frame take the form of

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 + T_1 \tag{16}$$

$$\begin{aligned} I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 + T_2 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 \end{aligned} \tag{17}$$

From the third equation, it is clear that $\dot{\omega}_3 = 0$. It is noted that once the symmetric axis is static, the spacecraft is not controllable using smooth control [29]. Therefore, we will assume the spacecraft that rotates constantly about its symmetric axis $[\omega_3(t) = \omega_{30} \neq 0]$ to render smooth nonlinear control laws. Besides the body-fixed reference frame, we also introduce an inertial reference frame $\mathbf{n} = (\hat{n}_1, \hat{n}_2, \hat{n}_3)$. The two reference frames are related by a rotation matrix that belongs to the special orthogonal group $SO(3)$. Consequently, the attitude trajectory of the moving reference frame with respect to the inertial reference frame can be described by a curve traced by the corresponding rotation matrix $R \in SO(3)$. $R(t)$ satisfies the following differential equation while moving along the trajectory:

$$\dot{R} = S(\omega_1, \omega_2, \omega_3)R$$

where $S(\omega_1, \omega_2, \omega_3)$ is a skew-symmetric matrix defined as

$$S(\omega_1, \omega_2, \omega_3) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

There are many ways to parameterize the rotation group, such as Euler angles, quaternion, Rodrigues parameters, etc. In this research, we will adopt a parameterization of $SO(3)$ from [18] to describe the relative orientation of two reference frames through two perpendicular rotations. Specifically, one rotation describes the location of the designated body axis (usually the spin axis of a spinning vehicle) in the inertial frame. Another one is an initial rotation about this axis. This formulation of the rotation matrix is very useful to the axisymmetric spacecraft control problem because the uniform revolution about its symmetric axis is ignorable. Moreover, the new parameterization will reduce the number of kinematic variables to 2. To describe the location of body 3-axis $\hat{b}_3$ in the inertial $\mathbf{n}$ frame, a stereographic projection is introduced with two kinematic parameters w_1 and w_2

$$w_1 = \frac{p_2}{1 + p_3} \qquad w_2 = \frac{-p_1}{1 + p_3}$$

where $-p_1$, $-p_2$, and p_3 are directional cosines of the axis $\hat{b}_3$ with respect to the $\mathbf{n}$ frame, that is, $\hat{b}_3 = -p_1\hat{n}_1 - p_2\hat{n}_2 + p_3\hat{n}_3$ with $p_1^2 + p_2^2 + p_3^2 = 1$. By construction $w_1 = w_2 = 0$ implies that the body 3-axis is aligned with the inertial 3-axis. Moreover, the stereographic projection is well defined for all orientations except for an upside-down configuration $(p_1, p_2, p_3) = (0, 0, -1)$ of the spacecraft. It is also easy to verify that its inverse map is given by

$$p_1 = \frac{-2w_2}{1 + w_1^2 + w_2^2} \qquad p_2 = \frac{2w_1}{1 + w_1^2 + w_2^2} \qquad p_3 = \frac{1 - w_1^2 - w_2^2}{1 + w_1^2 + w_2^2}$$

and can be used to determine p_1, p_2, and p_3 for the given w_1, w_2. Under this projection, w_1 and w_2 satisfy the differential equations

$$\dot{w}_1 = \omega_{30}w_2 + \omega_2 w_1 w_2 + \frac{\omega_1}{2}(1 + w_1^2 - w_2^2) \tag{18}$$

$$\dot{w}_2 = -\omega_{30}w_1 + \omega_1 w_1 w_2 + \frac{\omega_2}{2}(1 + w_2^2 - w_1^2) \tag{19}$$

Let

$$a = \frac{I_2 - I_3}{I_1}$$

assuming $-1 < a < 1$ for physical consideration,

$$m = \omega_{30} \qquad u_1 = \frac{T_1}{I_1} \qquad u_2 = \frac{T_2}{I_2}$$

and define the states as

$$[x_1 \quad x_2 \quad x_3 \quad x_4]^T = [\omega_1 \quad \omega_2 \quad w_1 \quad w_2]^T$$

a state-space model of axisymmetric spacecraft is then derived from Eqs. (16–19) as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & am & 0 & 0 \\ -am & 0 & 0 & 0 \\ \frac{1+x_3^2-x_4^2}{2} & x_3x_4 & 0 & m \\ x_3x_4 & \frac{1+x_4^2-x_3^2}{2} & -m & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{20}$$

For the rest of this paper, we will assume that $a = 0.5$, and $m = -0.5$ rad/s. It is noted that the spacecraft equation is in cascade form. That is, the control input u drives the angular velocity equation and the angular velocity ω drives the kinematic equation. The kinematic equation can only be accessed and manipulated through the angular velocity vector ω. To facilitate the application of the proposed nonlinear $\mathcal{H}_\infty$ control techniques to axisymmetric spacecraft control, we will also augment the spacecraft model (20) by adding some exogenous input. The external input used for the attitude stabilization problem will capture modeling uncertainty and actuator inaccuracy of the spacecraft. For spacecraft command tracking, external inputs will include modeling uncertainty, actuator and sensor inaccuracies, and command input.

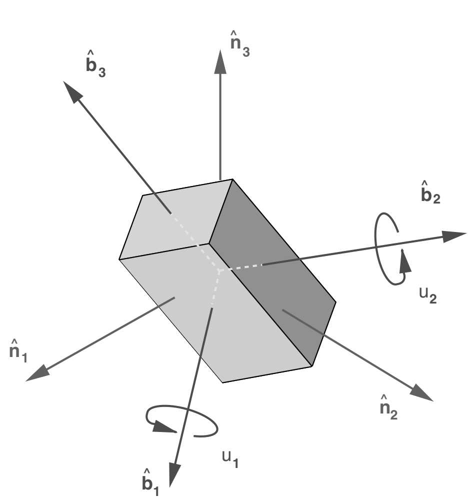


Fig. 1 Axisymmetric rigid body with two controls.

B. Nonlinear $\mathcal{H}_\infty$ Control Design and Simulation

As an underactuated system, it is well known that axisymmetric spacecraft with two control torques is generally not controllable. Nevertheless, feedback stabilization of such a spacecraft is achievable with respect to its symmetric axis. Indeed, a restricted attitude stabilization problem was solved for axisymmetric spacecraft in [29] using discontinuous nonlinear control. It was also shown that nonspinning dynamics cannot be asymptotically stabilized using smooth feedback. Tsiotras and Longuski [18] presented a new methodology of constructing linear and nonlinear feedback control laws for axisymmetric spacecraft using a backstepping technique. Time optimal nonsmooth feedback control designs for axisymmetric spacecraft have also been addressed in [28]. We are interested in solving attitude stabilization and command tracking problems for the axisymmetric spacecraft using computationally efficient nonlinear $\mathcal{H}_\infty$ control techniques. A key observation that makes rigid body control of spacecraft amenable to an efficient (convex) design technique such as SOS programming is that the spacecraft dynamic equation (20) represents its attitude dynamics by a polynomial vector field.

1. Spacecraft Attitude Stabilization

When $\omega_{30} \neq 0$, the steady-state motion of the axisymmetric spacecraft is a uniform revolution about its symmetric axis [18]. Therefore, the spacecraft attitude is expected to be stabilized on a two-dimensional plane instead of staying at an equilibrium point. The state-space model of spacecraft including disturbance d and controlled output variable e is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & am & 0 & 0 \\ -am & 0 & 0 & 0 \\ \frac{1+x_3^2-x_4^2}{2} & x_3x_4 & 0 & m \\ x_3x_4 & \frac{1+x_4^2-x_3^2}{2} & -m & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{21}$$

$$e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.8 & 0 \\ 0 & 0.8 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{22}$$

Note that the spacecraft model (21) and (22) does not satisfy orthonormal and decoupling assumptions. The disturbances d_1 and d_2 capture the modeling uncertainty and actuator calibration error, etc. We would like to design a nonlinear state feedback $\mathcal{H}_\infty$ controller to stabilize the spacecraft in the existence of external input. This corresponds to solving the first HJI inequality (3) in Theorem 1. For local control design, we introduce state region constraints $R_1(x) = x_3^2 + x_4^2 - 0.5 < 0$ and $R_2(x) = x_1^2 + x_2^2 - 0.5 < 0$. These constraints specify a bounded operating range of spacecraft angular velocity and attitude that the nonlinear $\mathcal{H}_\infty$ control is designed for. Consequently, the state feedback $\mathcal{H}_\infty$ control synthesis condition in optimization problem (12) will be modified to

$$\begin{aligned} &\min \gamma_i \\ &\text{subject to } U_i(x) + \lambda_1(x)R_1(x) + \lambda_2(x)R_2(x) \in \Phi_{\mathrm{SOS}} \\ &v^T\left\{-\mathcal{S}_{\mathbf{FIi}} + \begin{bmatrix} [\lambda_3(x) + \alpha x_1^4 + \beta x_2^4]R_1(x) + \lambda_4(x)R_2(x) & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} \end{bmatrix}\right\}v \\ &\in \Phi_{\mathrm{SOS}} \end{aligned} \tag{23}$$

where the SOS multipliers $\lambda_i(x) = x^T\Lambda_i x$ with $\Lambda_i > 0$, α and $\beta > 0$.

Using the iterative algorithm outlined in Sec. II.C, we solve the modified SOS problem (23) by SOSTOOL [13] and obtain a higher order Lyapunov function $U(x)$ as

$$\begin{aligned} \gamma &= 0.8943 \\ U(x) &= 46.2544x_1^2 + 8.4019x_2^2 + 152.8548x_3^2 + 60.7185x_4^2 \\ &\quad + 4.6060x_1x_2 + 98.0724x_1x_3 + 4.8102x_1x_4 + 4.7611x_2x_3 \\ &\quad + 11.9792x_2x_4 + 66.7882x_3x_4 + 19.4649x_1^4 + 4.5473x_2^4 \end{aligned}$$

The corresponding nonlinear state feedback control law is given by

$$\begin{aligned} u &= F(x) \\ &= \begin{bmatrix} -17.0455x_1 - 0.6534x_2 - 17.4743x_3 - 0.4122x_4 - 11.7462x_1^3 \\ -0.6534x_1 - 10.7249x_2 - 0.7326x_3 - 8.4464x_4 - 8.7985x_2^3 \end{bmatrix} \end{aligned}$$

which renders the closed-loop $\mathcal{H}_\infty$ norm less than $\gamma = 0.8943$. The domain of validity for this controller is the intersection of regions $R_1(x) < 0$ and $R_2(x) < 0$.

In case of zero disturbance, we choose an initial point $x_0 = [0.45 \quad 0.5 \quad -0.6 \quad 0.35]^T$ within the feasible state region. Figure 2 shows the state trajectories and the control inputs of the closed-loop system starting from the given initial condition. It is observed that evolved state trajectories satisfy state region constraints. Moreover, the closed-loop state converges to the origin, which implies the designed nonlinear control law stabilizes the spacecraft to a uniform revolution (circular attractor) rather than an isolated equilibrium.

As a comparison to the nonlinear $\mathcal{H}_\infty$ control approach, we will examine the backstepping control design of axisymmetric spacecraft developed in [18]:

$$\begin{aligned} u_1 &= -amx_2 + k\left[-mx_4 - \left(\frac{1 + x_3^2 - x_4^2}{2}\right)x_1 - x_3x_4x_2\right] \\ &\quad - (\phi + C_d)(x_1 + kx_3) \\ u_2 &= amx_1 + k\left[mx_3 - \left(\frac{1 + x_4^2 - x_3^2}{2}\right)x_2 - x_3x_4x_1\right] \\ &\quad - (\phi + C_d)(x_2 + kx_4) \end{aligned}$$

With $k = 20$, $\phi = 0.44$, and $C_d = 0.15$, we are able to achieve the same convergence rate as the nonlinear $\mathcal{H}_\infty$ design. C_d is a disturbance bound satisfying

$$\|B_1(x)d\| \leq C_d\left\|\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + k\begin{bmatrix} x_3 \\ x_4 \end{bmatrix}\right\| \tag{24}$$

The corresponding state trajectories and control inputs of the backstepping design are shown in Fig. 3. It is observed that our

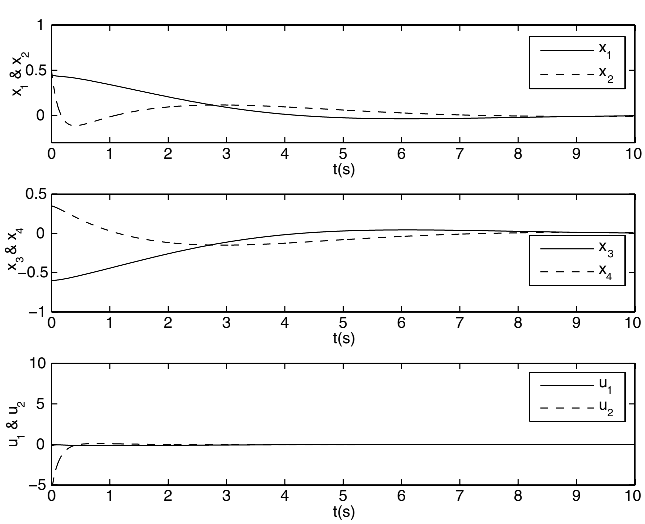


Fig. 2 Spacecraft stabilization using the nonlinear $\mathcal{H}_\infty$ control.

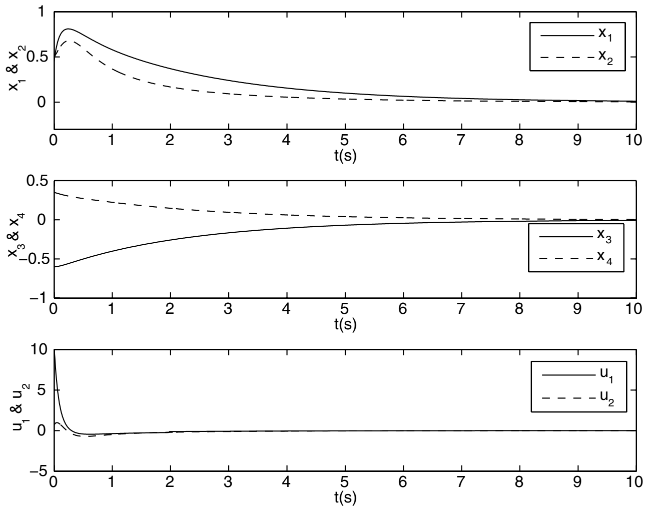


Fig. 3 Spacecraft stabilization using a robust backstepping control.

proposed state feedback control has less overshoot in states and requires smaller control efforts. This demonstrates the advantage of using proposed $\mathcal{H}_\infty$ control to achieve better spacecraft controlled attitude stabilization. The result is not surprising because the nonlinear $\mathcal{H}_\infty$ control is an optimized control design, whereas the backstepping control is only a feasible controller to stabilize the spacecraft.

Moreover, we would like to compare the disturbance rejection property of both controllers. Starting from the same initial condition, we choose a disturbance profile satisfying the constraint (24) as

$$\begin{cases} d_1(t) = 0.9x_1(t) + 4x_3(t) \\ d_2(t) = 0.5x_2(t) + 4x_4(t) \end{cases}$$

Such a disturbance depends on the variation of the states and converges to zero asymptotically, as required by the robust backstepping control design. Figure 4 shows the comparison of states and inputs using two types of control designs. The three figures on the left-hand side are the results using the state feedback nonlinear $\mathcal{H}_\infty$ control. Parallel results from the backstepping control are provided on the right. As expected, the backstepping control design guarantees the robust stability of the closed-loop system under external disturbances. The spacecraft eventually settles down to its equilibrium (i.e., uniform revolution about its symmetric axis). Using the state feedback nonlinear $\mathcal{H}_\infty$ control, the closed-loop system is also stabilized to the origin with better disturbance rejection performance. Moreover, the state region constraints are satisfied during the simulation. Different from the backstepping control design, the nonlinear $\mathcal{H}_\infty$ controller is able to accommodate persistent disturbances as well.

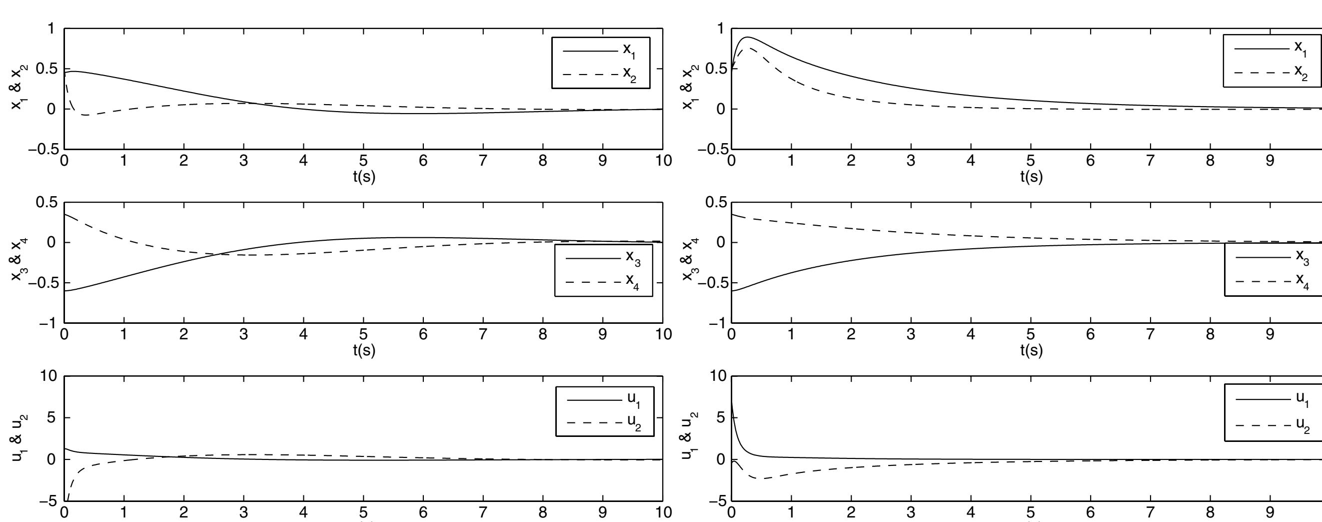
2. *Spacecraft Command Tracking*

To achieve accurate command tracking with output feedback $\mathcal{H}_\infty$ control, the control design needs to be modified from the attitude stabilization problem. Specifically, under the external disturbances such as modeling uncertainty, actuator and sensor inaccuracies, the spacecraft is desired to follow a command trajectory with tracking error as small as possible. The disturbance vector $[d_1 \quad d_2 \quad d_3 \quad d_4 \quad d_5]^T$ includes disturbances d_1 and d_2 along the $\hat{b}_1$ and $\hat{b}_2$ axes, output measurement noises d_3 and d_4, and the desired x_3 trajectory $x_{3d} = d_5$. The nonlinear control design objective is to minimize the effect of these disturbances on the tracking error $(x_3 - x_{3d})$ with reasonable control force.

To this end, we introduce a new state variable $z = \frac{\tau}{s}(x_3 - x_{3d})$ to eliminate the tracking error asymptotically. z reflects the accumulated effect of the tracking error over the past time and τ is a prespecified integration coefficient. By incorporating the dynamics of z into the original spacecraft model and redefining output variables, the augmented system will be

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & am & 0 & 0 & 0 \\ -am & 0 & 0 & 0 & 0 \\ \frac{1+x_3^2-x_4^2}{2} & x_3x_4 & 0 & m & 0 \\ x_3x_4 & \frac{1+x_4^2-x_3^2}{2} & -m & 0 & 0 \\ 0 & 0 & \tau & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ z \end{bmatrix} + \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\tau \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{25}$$

$$e = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ z \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{26}$$



a) Nonlinear $\mathcal{H}_\infty$ control **b) Backstepping control**

Fig. 4 Comparison of two control designs for spacecraft disturbance rejection.

$$y = \begin{bmatrix} 0 & 0 & 325.38 & 0 & 0 \\ 0 & 0 & 0 & 344.38 & 0 \\ 0 & 0 & 0 & 0 & 99.556 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ z \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix} \tag{27}$$

The new state variable is $\bar{x} = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad z]^T$. The output y includes measured attitudes w_1 and w_2 and accumulated tracking error z. Note that the augmented plant fits into the plant structure of Eq. (2) and satisfies the corresponding assumptions.

To solve the command tracking problem in output feedback form, we set $\tau = 0.1$ and choose $U(\bar{x})$ as a quadratic Lyapunov function and $V(\bar{x})$ as a higher order Lyapunov function. Similar to the attitude stabilization design, we specify the feasible state region in terms of spacecraft angular velocity and attitude as the intersection of $R_1(x) = x_3^2 + x_4^2 - 0.5 < 0$ and $R_2(x) = x_1^2 + x_2^2 - 0.5 < 0$. The local state feedback synthesis condition in Eq. (12) will then be modified by adding these state constraints. Moreover, the local output feedback synthesis condition in Eq. (14) needs to be modified to

$$\min \hat{\gamma}$$

$$\text{subject to } V(\bar{x}) + \lambda_1(\tilde{x})R_1(x) + \lambda_2(\tilde{x})R_2(x) \in \Phi_{\text{SOS}}$$

$$v_1^T \left\{ -\mathcal{S}_{\mathbf{FC}} + \begin{bmatrix} [\psi_1(\bar{x}) + \alpha x_1^4 + \beta x_2^4]R_1(x) + \psi_2(\bar{x})R_2(x) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\} v_1 \in \Phi_{\text{SOS}}$$

$$v_2 \left[-\frac{\partial^2}{\partial x^2} (\mathcal{H}_{FC} - \gamma^{-2}\mathcal{H}_{FI}) \right] v_2 \in \Phi_{\text{SOS}}$$

$$V(\bar{x}) - \gamma^{-2}U(\bar{x}) + \lambda_3(\tilde{x})R_1(x) + \lambda_4(\tilde{x})R_2(x) \in \Phi_{\text{SOS}}$$

where

$$\tilde{x} = [\bar{x}^T \quad x_1^2 \quad x_2^2 \quad x_1 x_2]^T \qquad \lambda_i(\tilde{x}) = \tilde{x}^T \Lambda_i \tilde{x}$$

$$\psi_j(\bar{x}) = \bar{x}^T \Psi_j \bar{x} \qquad \Lambda_i, \Psi_j > 0 \qquad \alpha, \beta > 0$$

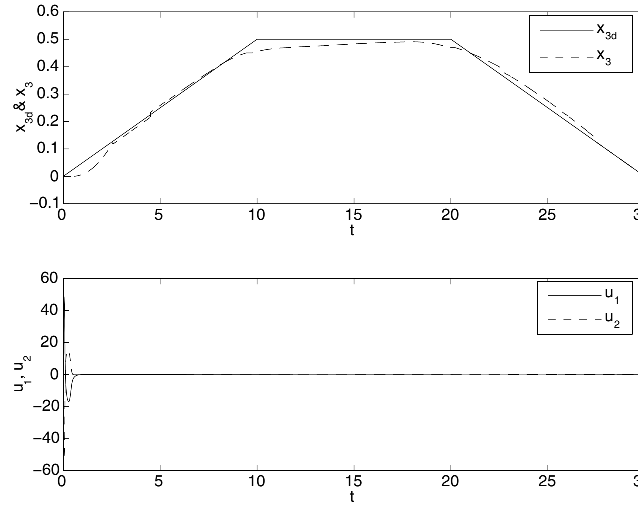


Fig. 5 Spacecraft tracking performance using the nonlinear $\mathcal{H}_\infty$ control.

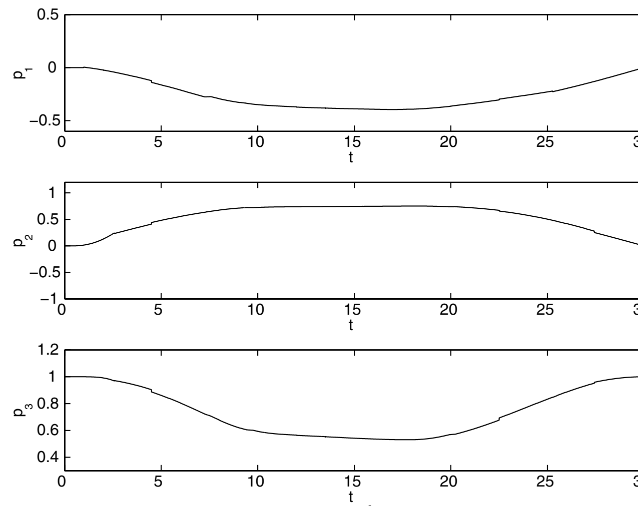
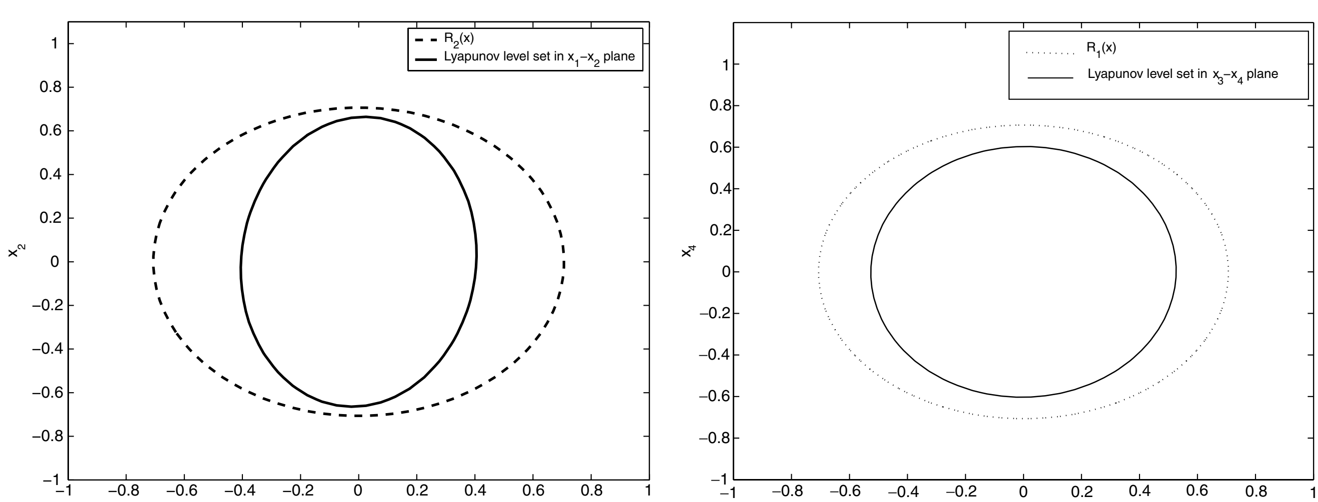


Fig. 6 Orientation of the body 3-axis $\hat{b}_3$ in terms of direction cosines.



a) In x_1-x_2 plane **b) In x_3-x_4 plane**

Fig. 7 Feasible region for nonlinear output feedback $\mathcal{H}_\infty$ control.

Applying the control design procedure in Sec. II.D with SOS programming, we obtain

$$\gamma = 0.827$$

$$U(\bar{x}) = 0.277x_1^2 + 0.284x_2^2 + 1.008x_3^2 + 0.886x_4^2 + 31.248z^2 - 0.182x_1x_2 + 0.918x_1x_3 - 0.582x_1x_4 + 4.593x_1z - 0.589x_2x_3 + 0.869x_2x_4 - 3.895x_2z - 1.481x_3x_4 + 10.469x_3z - 8.950x_4z$$

$$\hat{\gamma} = 0.605$$

$$V(\bar{x}) = 17.4502x_1^2 + 6.3302x_2^2 + 3603.5x_3^2 + 2745.3x_4^2 + 447.0119z^2 - 0.3061x_1x_2 - 434.3820x_1x_3 + 9.0606x_1x_4 + 11.6333x_2x_3 - 223.6766x_2x_4 - 64.9582x_3x_4 + 5.7611x_1^4 + 1.7748x_2^4 + 7.1536x_1^2x_2^2$$

After that, it is straightforward to calculate $F_0(\bar{x})$, $F_1(\bar{x})$, and $L_0(\bar{x})$ and construct the nonlinear command tracking control law in output feedback form (7). The synthesized controller is capable of tracking different types of command signals.

To examine tracking performance of the controlled spacecraft, we set d_1 to d_4 to be zero and x_{3d} as a trapezoidal profile. Starting from zero initial condition $\bar{x} = 0$, we obtain the simulation results as shown in Fig. 5. It is observed that good tracking of the given trajectory has been achieved. On the other hand, it is also possible to design a linear output feedback controller based on the linearized model of Eqs. (25–27) so that the closed-loop system has optimal $\mathcal{H}_\infty$ performance. However, the linear controller would not work well on the original nonlinear spacecraft for the given trajectory.

By an inverse map of the stereographic projection, Fig. 6 provides the orientation of the $\hat{b}_3$ axis in terms of cosines of three direction angles. It is observed that the body 3-axis initially coincides with the direction of $\hat{n}_3$ and is perpendicular to the $\hat{n}_1 - \hat{n}_2$ plane. As w_1 tracks the command trajectory, $\hat{b}_3$ rotates in three directions. Each direction cosine follows a similar trapezoidal profile and finally goes back to its original position.

The feasible region for the constructed nonlinear control law is generally a complicated domain in the four-dimensional state space. To visualize it, we project the feasible region onto $x_1 - x_2$ and $x_3 - x_4$ planes separately. Figure 7a shows the cross section of the resulting Lyapunov level set with the $x_1 - x_2$ plane and the state region constraint on x_1 and x_2 variables, whereas Fig. 7b describes the cross region in the $x_3 - x_4$ plane where the nonlinear $\mathcal{H}_\infty$ controller is feasible. As can be seen, all specified constraints $R_1(x) < 0$ and $R_2(x) < 0$ are satisfied.

IV. Conclusions

In this paper, we proposed an efficient $\mathcal{H}_\infty$ control design procedure for generalized nonlinear output feedback $\mathcal{H}_\infty$ control and developed nonlinear $\mathcal{H}_\infty$ control laws for axisymmetric spacecraft attitude stabilization and trajectory tracking problems. Because the spacecraft attitude dynamics is in the form of polynomial nonlinear systems, we are able to solve its nonlinear control problems using a SOS-based solution approach. In this framework, the difficult HJI inequalities associated with nonlinear $\mathcal{H}_\infty$ control synthesis are converted to polynomial matrix inequalities and solved using SOS programming techniques in polynomial time. The proposed control design approach is applicable to large classes of nonlinear systems with relaxed plant assumptions. Its effectiveness was demonstrated on an underactuated spacecraft control problem. In addition, higher order Lyapunov functions have been shown to be helpful in achieving better controlled performance for axisymmetric spacecraft with two control torques.

Acknowledgments

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